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Differentiability of the minimal average action as a function of the rotation number

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Abstract. Let $\bar{f}$ be a finite composition of exact twist diffeomorphisms. For any real number ω, let $A(\omega)$ denote the minimal average action of $\bar{f}$-invariant measures with angular rotation number ω. We prove that $A(\omega)$ is differentiable at every irrational number ω and that for generic $\bar{f}$ it is not differentiable at rational ω, thus verifying conjectures of S. Aubry. Moreover, we show that these results are valid for a variational principle h which satisfies the condition which we have called elsewhere (H). As a consequence, we generalize a result due to Bangert concerning geodesics on a two dimensional torus with an arbitrary, but sufficiently smooth metric.

Introduction

Let $\mathcal{T}_\beta^1$ denote the set of exact area preserving, orientation preserving, end preserving C^1 diffeomorphisms $\bar{f}$ of the infinite cylinder $(\mathbf{R}/\mathbf{Z}) \times \mathbf{R}$ which satisfy the positive twist condition with a lower bound of β on the minimum angle through which the vertical is turned (by $\bar{f}$ or $\bar{f}^{-1}$). (A precise definition of $\mathcal{T}_\beta^1$ is given in [4, §2], where it is called J_β, and repeated in [8, §1]). Let $\mathcal{P}_\beta^1$ denote the set of all diffeomorphisms of $(\mathbf{R}/\mathbf{Z}) \times \mathbf{R}$ which may be represented as finite compositions of elements of $\mathcal{T}_\beta^1$.

Consider $\bar{f} \in \mathcal{P}_\beta^1$. For each real number ω, there exists an $\bar{f}$- invariant probability measure μ_ω whose (angular) rotation number $\rho(\mu_\omega)$ is ω and which minimizes the average action $A(\mu)$ over $\bar{f}$-invariant probability measures μ of rotation number ω. We have given two proofs of this fact in [6] and [7, §6]. In fact, when ω is irrational, μ_ω is unique and is characterized by the fact that its support is the Aubry-Mather set of rotation number ω.

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Let $A(\omega) = A(\mu_\omega)$. The function $A(\omega)$ is strictly convex by a theorem of Aubry, so the left and right derivatives $A'(\omega-)$ and $A'(\omega+)$ exist. The main result of this note is that when ω is irrational, they are equal, i.e. A is differentiable at irrational ω. (I am indebted to S. Aubry for pointing out to me that this result and the discussion in §3, §4 is closely related to the discussion in [1, §5]. The relationship between our results and the conjectures of Aubry is discussed in §6.) We also obtain the estimate $A'(p/q+) - A'(p/q-) \leq 4\theta/q$, where $\theta = \cot\beta$. This estimate is valid when the rational number is expressed in lowest form as p/q, i.e. when p, q are relatively prime integers and $q > 0$.

The differentiability of $A(\omega)$ at irrational ω is an easy consequence of the following estimates

$$A'(p/q+) \leq q'q[A(p/q + 1/qq') - A(p/q)] \leq A'(p/q+) + 16\theta/q', \quad (1+)$$

$$A'(p/q-) \geq -q'q[A(p/q - 1/qq') - A(p/q)] \geq A'(p/q-) - 16\theta/q',. \quad (1-)$$

which hold when p/q is a rational number in lowest terms and q' is a positive integer. Of course, the first inequality in each line above follows immediately from the convexity of A. The main part of this note is a proof of the second inequality in each line above.

1. Proof of the inequalities (1±)

We will prove only the inequality (1+), the proof of the other inequality being similar. In fact, (1+) is simply (4.4a) of [5], corrected and suitably interpreted. We explain the appropriate interpretation below, and the correction in the Appendix.

Actually, the inequality (4.4a) of [5] holds more generally for a variational principle h which satisfies $(H_1) - (H_5)$ and $(H_{6\theta})$ of [4], as the proof in [5] shows. We will not restate these conditions here, but refer to [4] or [8, §1] for a statement of them. Note that conditions $(H_1) - (H_4)$ are stated in [2]. In [8, §1], we stated only conditions $(H_1), (H_2), (H_5)$, and $(H_{6\theta})$, since these conditions imply (H_3) and (H_4). We refer to the conditions $(H_1) - (H_5)$ and $(H_{6\theta})$ collectively as (H_θ).

The variational principle h associated to $\bar{f}$ satisfies (H_θ). We refer to [8, §1] for the detailed definition of h. Here, we just recall that if $\bar{f} = \bar{f}_k \circ \cdots \circ \bar{f}_1$, where

$\bar{f} \in \mathcal{T}^1$ then the variational principle h associated to $\bar{f}$ is the k-fold conjunction

$$h = h_1 * \cdots * h_k,$$

where h_i is the generating function of a lift f_i of $\bar{f}_i$ to the universal cover. We proved in [4] and [5] that h satisfies (H_θ); these results are summarized in [8].

Let $H(x, x') = h^{*q}(x, x' + p) - \text{const.}$, where $h^{*q} = h * \cdots * h$ (q times) denotes the q-fold conjunction of h with itself, and const. is chosen so that $\min H(x, x) = 0$. Let L be defined as in [5, §4], but using H in place of h. We have

$$qq'[A(p/q + 1/qq') - A(p/q)] = L.$$

The proof of this equation is simply a matter of untangling the definitions, which we leave to the reader. Since [5, **4.4a**] states that $K \leq L \leq K + 16\theta/q'$, we obtain $K = A'(p/q+)$, by letting $q' \to +\infty$. Then (1+) follows immediately. The definition of K which we use is given in §3. With the definition given in §3, the inequality [5, **4.4a**] is correct; with the definition given in [5], it is not correct. See also the Appendix to this paper.

This proof does not use the assumption that h is associated to an element of $\mathcal{P}^1$, only that h satisfies (H_θ). The only remark we need make is how to generalize $A(\omega)$ to this setting. In fact, when h is associated to $\bar{f} \in \mathcal{P}^1$, we have that $A(\omega)$ is the minimum of Percival's Lagrangian

$$P_\omega(\varphi) = \int_0^1 h(\varphi(t), \varphi(t+\omega))\, dt$$

taken over all bounded and measurable functions φ such that $\varphi(t+1) = \varphi(t) + 1$. Moreover, the minimizing function φ_ω is increasing. See [3], [9], and [6]. Thus, for general h which satisfy (H_θ), we will define $A(\omega)$ to be the minimum value of $P_\omega(\varphi)$. When it is necessary to explicitly indicate the dependence on h, we will write $A_h(\omega)$.

With this definition of $A_h(\omega)$, the estimates $(1\pm)$ still hold for any variational principle h which satisfies (H_θ).

2. Differentiability at irrationals

In the previous section, we showed that the estimates $(1\pm)$ hold when h satisfies the conditions (H_θ). Now we show that the estimate (1+) implies the differentiability of A at irrational numbers. (A similar argument would show that the estimate (1-) implies the differentiability of A at irrational numbers.) Let ω be irrational. By Dirichlet's pigeon hole principle, there exist rational numbers $p/q < \omega$ with $\omega - p/q < 1/q^2$ and with q arbitrarily large. Let q' be an integer such that $\omega < p/q + 1/qq'$.

From (1+) and the convexity of A, we may obtain an estimate for $A'(\omega+)$, as follows. Let P be the point $(p/q, A(p/q))$ in the plane, let $Q = (p/q + 1/qq', A(p/q + 1/qq'))$, let ℓ be the line through P with slope $A'(p/q+)$, let R be the point on ℓ whose first coordinate is ω, and let L be the line joining R and Q. Obviously, the slope of L is an upper bound for $A'(\omega+)$. From (1+), it follows easily that the slope of L is $\leq A'(p/q) + 16\theta/\lambda q'$, where λ is the ratio of $p/q + 1/qq' - \omega$ to $p/q + 1/qq' - p/q$, i.e. $\lambda = pq' + 1 - \omega qq'$. Using the convexity of A, we then obtain

$$A'(p/q+) \leq A'(\omega-) \leq A'(\omega+) \leq A'(p/q+) + 16\theta/\lambda q',$$

so $A'(\omega+) - A'(\omega-) \leq 16\theta/\lambda q'$.

Let $q' = [q/2]$. Since $p/q < \omega < p/q + 1/q^2$, we have $\lambda \geq 1/2$, so $0 \leq A'(\omega+) - A'(\omega-) \leq 32\theta/q'$. Since q' may be chosen to be arbitrarily large, by Dirichlet's pigeon hole principle, we obtain $A'(\omega+) = A'(\omega-)$.

3. Rational ω

In this section, we show that if $\omega = p/q$, in lowest terms, then $A'_h(p/q+) - A'_h(p/q-) \leq 4\theta/q$, provided that h satisfies (H_θ). The proof of this is based on the formula $A'(p/q+) = K$, given in §1, and the analogous formula for $A'(p/q-)$, which we will write in the form $A'(p/q-) = -K^-$. In this section, we will also write K^+ for K.

Next, we define $K^\pm$.

We let $\mathcal{A}$ denote the set of $x \in \mathbf{R}$ such that $H(x,x) = 0$, i.e. the set where $H(x,x)$ takes its minimum value. (This set was denoted A_0 in [5], but here we have used the symbol A for the average action.) For each complementary interval $J = [J^-, J^+]$ of $\mathcal{A}$, we define numbers K_J^+ and K_J^-, as follows. Choose an H-minimal configuration $(\ldots, x_i, \ldots)$ such that $x_i \to J^-$ (resp. J^+) as $i \to -\infty$

and $x_i \to J^+$ (resp. J^-) as $i \to +\infty$. Theorem 5.8 of [2] asserts the existence of such configurations. We set

$$K_J^+ \text{ (resp. } K_J^-) = \sum_{i=-\infty}^{\infty} H(x_i, x_{i+i}).$$

Thus, K_J^+ is what we called K_J in [5, §4]. We showed in [5, §4] that this sum is absolutely convergent and that

$$\begin{aligned}\int_{J^-}^{J^+} \partial_2 H(y, y+) dy &\leq K_J \\ &\leq H(J^-, J^+) \\ &= \int_{J^-}^{J^+} \partial_2 H(y, y+) dy + \mu_H(\Delta_J^+),\end{aligned}$$

where Δ_J^+ is the triangle $\{(y, z) : J^- \leq y \leq z \leq J^+\}$. Likewise, using the results of [5], it is possible to show that

$$\begin{aligned}-\int_{J^-}^{J^+} \partial_2 H(y, y+) dy &\leq K_J^- \\ &\leq H(J^+, J^-) \\ &= -\int_{J^-}^{J^+} \partial_2 H(y, y+) dy + \mu_H(\Delta_J^-),\end{aligned}$$

where Δ_J^- is the triangle $\{(y, z) : J^- \leq z \leq y \leq J^+\}$. Using [5, 3.2], we then obtain

$$0 \leq K_J^- + K_J^+ \leq \mu_H(J^2) \leq (J^+ - J^-)\nu_H^2(J^-, J^+). \tag{2}$$

For any interval I whose endpoints are in $\mathcal{A}$, we set

$$K_I^{\pm} = \sum K_J^{\pm} + \int_{I \cap \mathcal{A}} \partial_2 H(y, y+) dy,$$

where the sum is taken over all complementary intervals J of $\mathcal{A}$ in I. In view of the periodicity property of H, the quantity $K_I^{\pm}$ is unchanged if we replace I by a translate of itself by an integer. As a consequence, $K_I^{\pm}$ is independent of I, for intervals I of length 1 with their endpoints in $\mathcal{A}$. We set $K^{\pm} = K_I^{\pm}$, where I is any interval of length 1 whose endpoints are in $\mathcal{A}$.

In §1, we showed that $A'(p/q+) = K^+$, as a consequence of [5, **4.4a**]. In the Appendix, we explain why [5, **4.4a**] is correct, with the definition of K given here. (We denoted K^+ by K in §1). A similar argument shows that $A'(p/q-) = -K^-$. From (2), we obtain

$$\begin{aligned} 0 &\le A'(p/q+) - A'(p/q-) \\ &= K^- + K^+ \\ &\le \sum_J (J^+ - J^-)\nu_H^2(J^-, J^+), \end{aligned} \tag{3}$$

where the sum is taken over all complementary intervals J of $\mathcal{A}$ in an interval I of length 1 whose endpoints are in $\mathcal{A}$.

Let $I_0 = [I_0^-, I_0^+]$ and $I_1 = [I_1^-, I_1^+]$ be two intervals whose endpoints are in $\mathcal{A}$. Let $\xi_0^\pm = (\dots, \xi_{0i}^\pm, \dots)$ and $\xi_1^\pm = (\dots, \xi_{1i}^\pm, \dots)$ be the h-minimal configurations of rotation symbol p/q such that $\xi_{\alpha 0}^\pm = I_\alpha^\pm, \alpha = 0, 1$. We will say that I_0 and I_1 are *companions* if there exist $i, j \in \mathbb{Z}$ such that

$$\xi_{00}^\pm = \xi_{1i}^\pm + j.$$

In other words, these two intervals are companions if ξ_1^- is a translate of ξ_0^- and ξ_1^+ is the same translate of ξ_0^+.

If I and I' are companions then $K_I^\pm = K_{I'}^\pm$. This follows directly from the definitions: for the case that I and I' are complementary intervals of $\mathcal{A}$ in $\mathbb{R}$, it is enough to express $K_I^\pm$ and $K_{I'}^\pm$ in terms of h (instead of H), and the equation $K_I^\pm = K_{I'}^\pm$ for general companions follows from this case.

We will say that an interval I is a *fundamental interval* if there exists an h-minimal configuration $\xi = (\dots, \xi_i, \dots)$ such that I is a complementary interval of the set $\{\xi_i + j\}$. Obviously $\{\xi_i + j\} \subset \mathcal{A}$ and each interval of length 1 with endpoints in $\{\xi_i + j\}$ is the union of q complementary intervals of $\{\xi_i + j\}$, and any two such intervals are companions. Thus, we obtain

$$K^\pm = qK_I^\pm,$$

when I is a fundamental interval. Furthermore, (3) implies

$$0 \le K^- + K^+ = q(K_I^- + K_I^+) \le q(I^+ - I^-)\nu_H^2(I^-, I^+),$$

where $I^- < I^+$ denote the endpoint of I.

Since each interval of length 1 with endpoints in $\{\xi_i + j\}$ is the union of q fundamental intervals, there are $[q/2]+1$ of them which have length $\leq 2/q$. In addition, since the ν_H^2-measure of any interval of length 1 is θ (cf. [5, §4]), there are $[q/2]+1$ fundamental intervals in any interval of length I which have ν_H^2-measure $\leq 2\theta/q$. Thus, among the q fundamental intervals in an interval of length 1, we have described two subsets, each with $> q/2$ elements. These two subsets must have a common member I, and for this common member we have

$$0 \leq K^- + K^+ \leq q(I^+ - I^-)\nu_H^2(I^-, I^+) \leq 4\theta/q.$$

Since $A'(p/q+) - A'(p/q-) = K^- + K^+$, we have proved

$$0 \leq A'(p/q+) - A'(p/q-) \leq 4\theta/q.$$

4. Generic diffeomorphisms

In this section, we will show that for a generic $\bar{f} \in \mathcal{P}^1$, the function A is not differentiable at any rational number. In fact, we will show that A is differentiable at p/q if and only if there exists an invariant circle which goes around the infinite cylinder which consists entirely of orbits of rotation symbol p/q. (Recall that such orbits are periodic of period q.) This statement clearly implies our statement about generic $\bar{f}$.

In fact, our argument works for any variational principle h which satisfies the condition (H). For such a variational principle, we will obtain that the associated function $A = A_h$ is differentiable at p/q if and only if $\mathbb{R}$ is the union of minimal configurations of rotation symbol p/q.

In the notation of §3, this means that A is differentiable at p/q if and only if $\mathcal{A} = \mathbb{R}$. We base our proof on the formula

$$A'(p/q+) - A'(p/q-) = K^+ - K^- = \sum_J (K_J^+ - K_J^-),$$

discussed in §3. The sum is taken over all complementary intervals J of $\mathcal{A}$ in an interval of length 1 having endpoints in $\mathcal{A}$. If $\mathcal{A} = \mathbb{R}$, there are no such complementary intervals, and the right side vanishes, since it is the empty sum.

On the other hand, if J is a complementary interval of $\mathcal{A}$ in $\mathbb{R}$, then $K_J^- +$

$K_J^+ > 0$. In fact,

$$K_J^+ > \int_{J-}^{J+} \partial_2 H(y, y+)\, dy$$

and

$$K_J^- > -\int_{J-}^{J+} \partial_2 H(y, y+)\, dy.$$

For the case of K_J^+, we proved this inequality in [5, §4], although we stated it with the relation $\geq$ rather than $>$, since that was all we needed there. Our proof there was based on [5, 3.4] (for infinite sums), and we have the strict inequality, since both the first and third terms on the right side of [5, 3.4] are positive in our situation. The proof for the case of K_J^- is the same.

5. Bangert's Results

In the abstract of this paper, we mentioned that our results generalize a result which Bangert told us about. Bangert's result concerns geodesics on a two dimensional torus provided with an arbitrary, but sufficiently smooth Riemannian metric. To such a metric, one may associate a variational principle h which satisfies the condition (H). The differentiability result obtained in previous sections for $A_h(\omega)$ implies the result of Bangert which we have referred to. Bangert obtained his result by a different method, which has independent interest.

In this section, we state Bangert's result and prove it by our method.

We let $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ denote the torus. We consider any C^4 Riemannian metric on T^2. For any $u \in H_1(T^2, \mathbf{Z})$, let ℓ_u denote the length of a shortest closed curve on T^2 representing u. It is well known that such a curve exists and is a geodesic. Bangert [2] showed that there exists a unique norm $\|\ \|$ on $H_1(T^2, \mathbf{R})$ such that $\ell_u = \|u\|$, for every $u \in H_1(T^2, \mathbf{Z})$. Let B be the unit ball of Bangert's norm. Let x be an element of the boundary ∂B of B and let $[x]$ denote the one dimensional subspace of $H_1(T^2, \mathbf{R})$ which contains x. Bangert's result which we have referred to is that ∂B has a tangent at x, provided that $[x] \cap H_1(T^2, \mathbf{Z}) = 0$.

We may prove this, as follows. Choose a basis v, w of $H_1(T^2, \mathbf{Z})$, so that every element of $H_1(T^2, \mathbf{Z})$ can be represented uniquely as a linear combination

$mv+nw$, with $n, m \in \mathbf{Z}$. Let γ_v be a closed curve which represents the homology class v and has minimal length. Let $\tilde{\gamma}_v$ be a lift of v to the universal cover $\mathbf{R}^2$ of the torus and let T_w be the Deck transformation of the torus which corresponds to w.

Choose some C^2 diffeomorphism $\varphi : \mathbf{R}/\mathbf{Z} \to \gamma_v$ and let $\tilde{\varphi} : \mathbf{R} \to \tilde{\gamma}_v$ be a lift of it. For any $x, x' \in \mathbf{R}$, let $h(x, x')$ denote the distance between $\tilde{\varphi}(x)$ and $T_w\tilde{\varphi}(x')$. Here, the distance is measured by the lift to $\mathbf{R}^2$ of the given metric on T^2. The function h satisfies the conditions (H). We leave the verification to the reader. It is easily verified that

$$A_h(\omega)^{-1}(w + \omega v) \in \partial B.$$

Thus, as we vary ω from $-\infty$ to ∞, we obtain a parameterization of $\partial B \cap \mathcal{H}$, where $\mathcal{H}$ is the half plane

$$\mathcal{H} = \{\alpha v + \beta w : \beta > 0\}$$

in $H_1(T^2, \mathbf{R})$

Bangert's theorem that ∂B has a tangent at x, provided that $[x] \cap H_1(T^2, \mathbf{Z}) = 0$, follows immediately.

6. Aubry's Discussion

There is a close connection with the results presented in this paper and the discussion in [1, §5], as S. Aubry has pointed out in a letter to me. The part of his letter which is relevant to this note follows: "There is *no rigorous proof* in [1,§5] but only 'physicists' ideas' and thus many conjectures. However, I think that some of them could be turned into rigorous statements."

"The function 'average action' $A(\omega)$ which you consider is called in my paper 'energy per atom' $\psi(\ell)$ and then $\omega = \ell/2a$. Since $\psi(\ell)$ is a convex function, the right derivative $\psi'^{+}(\ell)$ and the left derivative $\psi'^{-}(\ell)$ of $\psi(\ell)$ are both defined. In this work, I have been especially interested in the inverse function $\ell(\mu)$ implicitly defined by the inequality $\psi'^{-}(\ell) \leq \mu \leq \psi'^{+}(\ell)$. Although the results were not presented in mathematical terms, they implicitly suggested several mathematical conjectures for this curve $\ell(\mu)$ (which I called the Devil's Staircase) which are

1. "For each rational number r/s $\ell(\mu)$ has a constant plateau (where $\ell(\mu) = 2ar/s$). This plateau is determined by $\psi'^{-}(2ar/s) \leq \mu \leq \psi'^{+}(2ar/s)$. The width $\psi'^{+}(2ar/s) - \psi'^{-}(2ar/s)$ is strictly non-zero unless the periodic cycle is continuously non-degenerate on an invariant circle. (The fact that this width is zero is equivalent to saying either of the following: a) the Peierls-Nabarro-barrier of the commensurate structure is zero, or b) the energy difference between the advanced and delayed discommensurations is zero, as was discussed in [1]).

2. "For irrational $\omega = \ell/2a$, we expected that $\psi'^{-}(\ell) = \psi'^{+}(\ell)$ because the defect energies go to zero at the incommensurate limit.

"In fact, these two statements were well known and well admitted in physics for the theory of incommensurate structures although no rigorous statement for any model existed. In one of the preprints you sent me, you gave a proof of these two conjectures for the twist map."

The last sentence refers to a preliminary version of this note. Conjecture 1 is equivalent to the statement that "A is differentiable at p/q if and only if there exists an invariant circle which goes around the infinite cylinder which consists entirely of orbits of rotation symbol p/q", which we proved in §4. Conjecture 2 is equivalent to the differentiability of A at irrational numbers, i.e. the main result of this note.

Appendix.

Errata to [5]. As noted above, the definition of K given in [5, §4] has to be changed in order for the inequalities [5, 4.4a] to be correct. Likewise, the definition of $K(\xi)$ given in [5, §4] has to be changed in order for the inequalities [5, 4.4b] to be correct. We have already given the correct definition of K in §3, but we repeat it here.

In this Appendix, we use the notation from [5]. Thus, A_0 denotes the set of $x \in \mathbf{R}$ for which $h(x,x)$ takes its minimum value (which is 0 by normalization). (A_0 and h were denoted $\mathcal{A}$ and H in §3.) We let I be an interval of length 1 with endpoints in A_0. The correct definitions are:

$$K = \sum_J K_J + \int_{I\cap A_0} \partial_2 h(x, x+)dx,$$

$$K(\xi) = \sum_J K_J(\xi) + \int_{I\cap A_0} \partial_2 h(x, x+)dx,$$

where in each case the sum is taken over all complementary intervals J of A_0 in I.

The first error in [5] is the inequality $\sum_J L_J \leq L$, which was asserted in [5, §4]. The correct inequality is

$$\sum_J L_J + \int_{I\cap A_0} \partial_2 h(x, x+)dx \leq L.$$

The argument given in [5, §4] is, of course, erroneous for the inequality $\sum_J L_J \leq L$: the error is the statement that "the contribution of the second term [of the right side of [5, 3.4] to $\sum_J L_J$] is $\int_x^{x+1} \partial_2 h(y, y+)dy$", but this becomes correct if $\sum_J L_J$ is replaced by $\sum_J L_J + \int_{I\cap A_0} \partial_2 h(x, x+)dx$.

The second error is the formula

$$K = \sum_{y\in B} h(y, y) + \int_x^{x+1} \partial_2 h(y, y+)dy + \mu_h(\Delta_B)$$

in [5, §4], which, however, is correct (by [5, 3.4]) for the definition of K which we have given here, although it is not correct for the definition of K given in [5, §4].

The principle results of [5] are deduced from [5, 4.4]. However, all that is used is that there exist real numbers K and $K(\xi)$ such that [5, 4.4] and [5, 4.2] hold. But [5, 4.2], i.e. $P_{0+}(\xi) = K(\xi) - K$ is unchanged by the new definitions, since we have added the same quantity to $K(\xi)$ as to K in changing the definitions. Thus, the proof given in [5, §4] of [5, 4.2] is still valid with the new definition.

Finally, we note the following misprint in the proof of [5, 4.2]: in the expression given on page 208 of [5] for $G_{0+}(x)$, the quantity K should be replaced by

K_J, where J is the complementary interval of A_0 which contains ξ.

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